Hamilton-Jacobi Tunneling Method for Dynamical Horizons in Different Coordinate Gauges

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Abstract

Previous work on dynamical black hole instability is further elucidated within the Hamilton-Jacobi method for horizon tunneling and the reconstruction of the classical action by means of the null-expansion method. Everything is based on two natural requirements, namely that the tunneling rate is an observable and therefore it must be based on invariantly defined quantities, and that coordinate systems which do not cover the horizon should not be admitted. These simple observations can help to clarify some ambiguities, like the doubling of the temperature occurring in the static case when using singular coordinates and the role, if any, of the temporal contribution of the action to the emission rate. The formalism is also applied to FRW cosmological models, where it is observed that it predicts the positivity of the temperature naturally, without further assumptions on the sign of energy.

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1 Introduction

In previous papers [1, 2] we considered the quantum instability of dynamical black hole using a variant of the tunneling method introduced by Parikh and Wilczek (PW) in the static case [3, 4] to uncover aspects of the back reaction effects; see also [5]. The PW tunneling method was refined and extended to more general cases in [6] and other papers [7, 8], including its relation with thermodynamics [9]. Other points of view and criticisms can be found in [10].

In the present paper, namely in section 2, we review this variant method, which in the static case goes under the name of Hamilton-Jacobi (HJ) method of tunneling, and was introduced in [11, 12]. A complete comparison analysis with the Parikh-Wilczek method has been presented in [13]. The HJ tunneling (across a horizon) connects the invariant surface gravity to a dynamical local temperature through the leading term in the black hole tunneling rate (I.1).

First, we would like to notice that the main novelty of the HJ method is its manifest covariance, compared to the original Parikh-Wilzcek approach [3, 4] where a non manifestly covariant Hamiltonian formulation was used. With regard to this issue, it is our opinion that HJ method is particularly suitable for the generalization to the dynamical spherically symmetric case, as we will try to substantiate in this paper.

The tunneling method provides not only new physical insight to an understanding of the black hole radiation, but is also a powerful and simple way to arrive at an expression for the surface gravity for a vast range of solutions, especially non-stationary and, in our case, dynamical black holes. This looks important since several definitions of the surface gravity for evolving horizons have been proposed in the past, all fitting some kind of first law of black hole mechanics more or less equally well. Still different results are advocated for expanding cosmological black holes in [14, 15]. A comparison is discussed thoroughly in [16, 17], but we anticipate that Hayward's definition [18] of surface gravity, related to Kodama's theory [19] of spherically symmetric space-times, is the most interesting to us since it is the one to which the tunneling method naturally leads.

As we said above, a way to understand Hawking radiation is by means of tunneling of particles through black-hole horizons. Such a tunneling approach uses the fact that the WKB approximation of the tunneling probability for the classically forbidden trajectory from inside to outside the horizon is:

$$\Gamma \propto \exp\left(-\frac{2}{\hbar}\text{Im }I\right), \tag{I.1}$$

where I is the classical action of the (massless) particle, to leading order in $\hbar$. It is of the utmost importance that the exponent be a scalar invariant, otherwise no physical meaning can be given to Γ. If, in particular, it has the form of a thermal emission spectrum with $2\text{Im }I = \beta\omega$, then the inverse temperature β and the particle's energy ω have to be separately scalars, since otherwise no invariant meaning can be given to the horizon temperature, which would not be an observable. If, even in the presence of reasonable physical

conditions, more than one prescription for defining an invariant energy is available, then also more notions of invariant temperature will exist and further analysis, or observations, could be needed. In most cases this could only be a scale transformation or a choice of a different family of observers.

In principle, all the Standard Model particles are expected in the Hawking radiation spectrum. However, most of the calculations in the literature have been performed just for scalar fields. Spin one-half emission was considered in [20] for stationary black holes, in [21] for the special case of the BTZ black hole and finally [22] studied the case of evolving horizons.

In Ref. [2] we have already discussed the Hamilton-Jacobi tunneling method for arbitrary spherically symmetric dynamical black holes. In this paper, we would like to present a systematic derivation of the results, pointing out the key points and making use of a covariant coordinate approach[1] which is particularly convenient in the discussion of several explicit examples we will consider as applications of the general method. We point out that our description of the tunneling rate is not equivalent to the "canonically invariant" description due to Chowdhury [24], which in fact obtains half the correct temperature for black holes and infalling shells. Probably a comparison analysis would be worthy, at this point.

Recall that any spherically symmetric metric can locally be expressed in the form

$$ds^2 = \gamma_{ij}(x^i)dx^i dx^j + R^2(x^i)d\Omega^2, \qquad i,j \in \{0,1\}, \tag{I.2}$$

where the two-dimensional metric

$$d\gamma^2 = \gamma_{ij}(x^i)dx^i dx^j \tag{I.3}$$

is referred to as the normal metric, x^i are associated coordinates and $R(x^i)$ is the areal radius, considered as a scalar field in the normal two-dimensional space. Another relevant scalar quantity on this normal space is

$$\chi(x) = \gamma^{ij}(x)\partial_i R \partial_j R. \tag{I.4}$$

The dynamical trapping horizon, say H, may be defined by

$$\chi(x)\Big|_H = 0, \qquad \partial_i \chi\Big|_H \neq 0. \tag{I.5}$$

This is equivalent to the vanishing of the expansion $\theta_{(\ell)}$ of the null, future directed congruence which is normal to a section of the horizon. In general there will be another null congruence of "incoming rays" with a related expansion $\theta_{(n)}$: if $\theta_{(n)} < 0$ and also $\mathcal{L}_n \theta_{(\ell)} < 0$ along H, then the horizon is of the future, outer type. Most of our results will be valid for this type of trapping horizon.

The Misner-Sharp gravitational mass, in units $G = 1$, is defined by

$$m(x) = \frac{1}{2}R(x)\left(1 - \chi(x)\right). \tag{I.6}$$

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[1]For static black holes a coordinate-free formulation has been given by Stotyn and co-workers [23].

This is an invariant quantity on the normal space. Note also that, on the horizon, $m|_H = m = R_H/2$. Furthermore, one can introduce a dynamic surface gravity [2] associated with this dynamical horizon, given by the normal-space scalar

$$\kappa_H = \frac{1}{2}\Box_\gamma R\Big|_H = \frac{1}{2\sqrt{-\gamma}}\partial_i(\sqrt{-\gamma}\gamma^{ij}\partial_j R)\Big|_H . \tag{I.7}$$

Recall that, in the spherical symmetric dynamical case, it is possible to introduce the Kodama vector field K, such that $(K^\alpha G_{\alpha\beta})^{;\beta} = 0$ that can be taken as its defining property. Given the metric (I.2), the Kodama vector components are

$$K^i(x) = \frac{1}{\sqrt{-\gamma}}\varepsilon^{ij}\partial_j R , \qquad K^\theta = 0 = K^\varphi . \tag{I.8}$$

The Kodama vector gives a preferred flow of time and in this sense it generalizes the flow of time given by the static Killing vector in the static case. As a consequence, we may introduce the invariant energy associated with a particle by means of the scalar quantity on the normal space

$$\omega = -K^i\partial_i I , \tag{I.9}$$

where I is the classical action of the massless particle, which we assume to satisfy the Hamilton-Jacobi equation

$$\gamma^{ij}\partial_i I\partial_j I = 0 . \tag{I.10}$$

We stressed above the importance to have at disposal an invariant definition of energy. Eq. (I.9) certainly satisfies this requirement if the action is, as it normally is, a scalar. In the following our aim will be not so much a detailed picture of the physical process of horizon tunneling, as to show that there is a precise invariant prescription to deal with the imaginary part of the action, in case there is one, which is valid for all solutions in all coordinates systems which are regular across the horizon.

The plan of the paper is the following. In Section II we discuss the role of coordinate invariance in the Hamilton-Jacobi method. In Section III, the first law for dynamical black holes is derived. In Section IV, FRW space-times are investigated in detail as relevant examples of dynamical horizons. The naturally obtained positivity of the temperature is emphasized. In Section V, static black holes are revisited, and the issue of Hayward surface gravity and Killing surface gravity is discussed in detail. The paper ends with the Conclusions.

2   Hamilton-Jacobi Tunneling and Gauge Invariance

To begin with, we would like to point out that, within this context, the tunneling has to be interpreted in a different way from the usual textbook treatment of tunneling effect in (non-relativistic) quantum mechanics. In fact, strictly speaking, there is no barrier here:

an issue which has been clarified by Parikh and dubbed by him as "secret tunneling" [4]. Furthermore, the appearance of an imaginary part in the "classical" action is due to the presence of a dynamical horizon and the use of Feynman's $i\epsilon$–prescription in dealing with an otherwise divergent integral within the classical realm. This is the way, together with equation (I.1), in which quantum effects enter into the game.

To be more specific, our key assumption is the following: that we can reconstruct the whole action I, from $\partial_i I$ by means of

$$I = \int_\gamma dx^i \partial_i I \,, \tag{II.1}$$

where γ is an oriented, *null curve*, with at least one point on the dynamical horizon. We can split the integration along this null curve in two pieces, one very near to the dynamical horizon, the remaining contribution living in the regular part of space-time. Then, we may perform a near-horizon approximation in the first integral. As a consequence, we have to use a system of coordinates which is regular on the horizon, otherwise this procedure *cannot* be applied, due to highly singular quantities involved. This procedure permits to evaluate the integration along what we can call, certainly with some abuse of notation, the "radial" and "temporal" parts and see if an imaginary part shows up, and what is their relation.

In Ref. [2], we have presented a derivation of the relevant imaginary part of the classical action and tunneling rate (I.1), valid for a future trapped horizon, which reads

$$\text{Im } I = \text{Im } \left(\int_\gamma dx^i \partial_i I \right) = \frac{\pi \omega_H}{\kappa_H} \,, \tag{II.2}$$

where κ_H is the dynamical surface gravity (I.7), and ω_H the Kodama energy evaluated on the dynamical horizon. These quantities are scalars in the normal space, thus the leading term of the tunneling rate is invariant, as we expect to be from an observable.
For the sake of completeness and clarity, we are briefly reporting the essential steps of the derivation in several coordinate systems.

The EFB Gauge. Let us start with giving the computation as in [2]. The key point is the observation that it is always possible to rewrite, locally, any spherically symmetric metric in an Eddington-Finkelstein-Bardeen (EFB) form which is regular on a trapping horizon. For black holes, one is concerned with future trapping horizons and so the advanced (rather than retarded) form is used:

$$ds^2 = -e^{2\Psi} C dv^2 + 2e^\Psi dv dr + r^2 d\Omega^2 \,, \tag{II.3}$$

where we are using $x^i = (v, r)$ as coordinates, and $C = C(v, r)$, $\Psi = \Psi(v, r)$. In this gauge, life is easy since the areal radius of the spheres of symmetry R turns out to be equal to the r-coordinate, i.e. $R = r$, and $\chi = C$: the horizon location is defined by the condition $C(v, r)|_H = 0$. The Kodama vector and the invariant energy assume the simple expressions given by $K = (e^{-\Psi}, \overrightarrow{0})$, and $\omega = -e^{-\Psi} \partial_v I$ while the invariant surface gravity is just given

by $\kappa_H = \partial_r C_H / 2$. One may note that Ψ transforms as an ordinary Liouville field, i.e. $\Psi \to \Psi + \ln |\partial \tilde{v} / \partial v|$, under $v \to \tilde{v}(v)$, making ω invariant under reparametrizations of the advanced time coordinate.

Expanding along a null direction from a point on H outward in a neighbourhood of the horizon gives (since $C_H = 0$)

$$0 = e^{\Psi_H} \Delta r \Delta v \,, \tag{II.4}$$

which shows that the temporal (v-)contribution does not play any role in what concerns us at the moment (the evaluation of Im I).

From the HJ equation, we see that, for outgoing modes,

$$C(\partial_r I) = 2\omega \,. \tag{II.5}$$

Thus, taking (II.4) and (II.5) into account, one has

$$
\begin{aligned}
\mathrm{Im}\, I &= \mathrm{Im} \int_\gamma (\partial_r I dr + \partial_v I dv) \\
&= \mathrm{Im} \int_\gamma dr\, \frac{2\omega}{C} \tag{II.6} \\
&= 2\,\mathrm{Im} \int_\gamma dr\, \frac{\omega}{\partial_r C \big|_H (r - r_H - i0)} \tag{II.7} \\
&= \frac{\pi \omega_H}{\kappa_H} \,, \tag{II.8}
\end{aligned}
$$

where again, the quantity C has been expanded around the horizon along the null direction, that is

$$C(v, r) \approx \partial_r C \big|_H \Delta r + \ldots \tag{II.9}$$

and Feynman's $i\epsilon$-prescription has been implemented in order to deal with the simple pole (II.7). $\kappa_H = \partial_r C\big|_H / 2$ coincides with our geometrical expectations and we see that, in EFB coordinate system, the temporal integration does not give any contribution to the imaginary part of the action of particles tunneling through the (trapping) horizon.

However, for practical reasons, it might be convenient to work in other (regular on the horizon) coordinate systems. So, denoting the temporal and spatial coordinates by $x^i = (t, r)$, we are going to discuss three more instructive gauges.

The r-Gauge. The normal metric here is non-diagonal, but as in EFB gauge, $R = r$. We have

$$ds^2 = d\gamma^2 + r^2 d\Omega^2 \,, \tag{II.10}$$

where the reduced normal metric is

$$d\gamma^2 = -E(r, t)dt^2 + 2F(r, t)dtdr + G(r, t)dr^2 \,, \qquad F \neq 0 \,. \tag{II.11}$$

The horizon is located where

$$\chi(t,r) = \gamma^{ij}\partial_i R \partial_j R = \gamma^{rr}(t,r) = \frac{E}{EG + F^2} \tag{II.12}$$

vanishes, i.e. at $E_H = 0$, provided $F_H \neq 0$. The Kodama vector reads

$$K = \left(\frac{1}{\sqrt{F^2 + EG}}, \overrightarrow{0}\right), \tag{II.13}$$

and the invariant energy

$$\omega = -\frac{\partial_t I}{\sqrt{F^2 + EG}}. \tag{II.14}$$

The dynamical surface gravity is, from (I.7),

$$\kappa_H = \left[\frac{1}{2F^3}\left(E'F - \frac{1}{2}\dot{E}G\right)\right]_H, \tag{II.15}$$

where an overdot and a prime denote differentiation with respect to t and r, respectively. From the metric the null, radial, expansion gives $\Delta t = -\frac{G}{2F}\Big|_H \Delta r$ (the other solution being related to the ingoing null ray), so within the near-horizon approximation and after some calculation we get $\chi \simeq 2\kappa_H(r - r_H)$; also, $\partial_t I = -F_H \omega$ from definition (II.14) and the horizon condition $E = 0$.

Splitting the integration along γ according to what we said above, we end with I given by the sum of a real term and a possibly imaginary part coming from near horizon approximation:

$$I \int_\gamma (dr\partial_r I + dt\partial_t I) = \int_\gamma dr \left[\partial_r I + \frac{1}{2}G_H\omega_H\right]. \tag{II.16}$$

What is remarkable is that in this gauge, the temporal, t–part is present[2], but being regular, it does not contribute to the imaginary part of the action. Making use of the HJ equation, the Kodama energy expression (II.14) and equation (II.12) as well, one has

$$\chi(\partial_r I)^2 - 2\frac{\omega F}{\sqrt{EG + F^2}}\partial_r I - \omega^2 G = 0, \tag{II.17}$$

thus, for outgoing modes,

$$\partial_r I = \frac{\omega F}{\sqrt{EG + F^2}}\frac{1}{\chi}\left(2 + O(\chi)\right). \tag{II.18}$$

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[2]Since it contributes to the total action through the $\frac{1}{2}(G\omega)_H$ term.

Making use of this equation and Feynman's prescription, $E_H = 0$ and $\chi \simeq 2\kappa_H(r - r_H)$, one has for the outgoing mode

$$
\begin{aligned}
\mathrm{Im}\ I \quad &= \mathrm{Im}\ \int_\gamma dr\, \partial_r I \\
&= \mathrm{Im}\ \int_\gamma dr\, \frac{\omega F}{\sqrt{F^2 + EG}} \frac{1 + \sqrt{1 + O(\chi)}}{2\,\kappa_H} \frac{1}{(r - r_H - i0)} \\
&= \frac{\pi \omega_H}{\kappa_H} \,,
\end{aligned}
\tag{II.19}
$$

in agreement with the EFB gauge.

The Synchronous Gauge. The second coordinate system we would like to consider is described by the line element

$$
ds^2 = -dt^2 + \frac{1}{B(r,t)} dr^2 + R^2(r,t)d\Omega^2 = d\gamma^2 + R^2(r,t)d\Omega^2 \,,
\tag{II.20}
$$

in which the metric is diagonal, but the proper radius of the spheres of symmetry R is a function of the coordinates r and t. In this case, one has

$$
\chi = -(\partial_t R)^2 + B(\partial_r R)^2 \,,
\tag{II.21}
$$

thus the future sheet of the trapping horizon $\chi_H = 0$ is given by

$$
(\partial_t R)_H = -\sqrt{B_H}(\partial_r R)_H \,,
\tag{II.22}
$$

in which we are assuming again a regular coordinate system on the horizon, namely that B_H and its partial derivatives are non-vanishing. The Kodama vector reads

$$
K = (\sqrt{B}\partial_r R, -\sqrt{B}\partial_t R, 0, 0) \,,
\tag{II.23}
$$

and the invariant energy

$$
\omega = \sqrt{B}(-\partial_r R \partial_t I + \partial_t R \partial_r I) \,.
\tag{II.24}
$$

The dynamical surface gravity may be evaluated and reads

$$
\kappa_H = \frac{1}{4}\left(-2\partial_t^2 R_H + 2B_H \partial_r^2 R_H + \frac{1}{B_H}\partial_t R_H \partial_t B_H + \partial_r R_H \partial_r B_H\right) \,.
\tag{II.25}
$$

Making use of the horizon condition, we may rewrite

$$
\kappa_H = \frac{1}{4}\left(-2\partial_t^2 R_H + 2B_H \partial_r^2 R_H - \frac{1}{\sqrt{B_H}}\partial_r R_H \partial_t B_H + \partial_r R_H \partial_r B_H\right) \,.
\tag{II.26}
$$

From the metric, the HJ equation reads simply

$$
-(\partial_t I)^2 + B(\partial_r I)^2 = 0 \,.
\tag{II.27}
$$

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As a consequence, the outgoing temporal contribution is equal to the radial one and we have

$$I = 2 \int_\gamma dr \partial_r I \, . \tag{II.28}$$

The HJ equation and the expression for the invariant energy lead to

$$\partial_r I = \frac{\omega}{B \partial_r R + \sqrt{B} \partial_t R} \, . \tag{II.29}$$

which has a pole on H. Making the expansion along the outgoing null curve, for which $\Delta t = -\frac{1}{\sqrt{B_H}} \Delta r$, in the near-horizon approximation, one gets

$$\text{Im } I = 2 \cdot \text{Im } \int_\gamma dr \frac{\omega}{2\kappa_H(r - r_H - i0)} = \frac{\pi \omega_H}{\kappa_H} \, , \tag{II.30}$$

which again coincides with the previous result. But notice that in this gauge, the temporal contribution is essential indeed to provide the correct result: without it the temperature would be doubled.

Conformal 2D Gauge. Another coordinate system where the temporal contribution to the action plays an essential role is the general diagonal form of a spherically symmetric metric, which reads

$$ds^2 = e^{\psi(t,r)} \left(-dt^2 + dr^2 \right) + R^2(t,r) d\Omega^2 \, . \tag{II.31}$$

In this form, the normal metric is conformally related to the 2-dimensional Minkowski space-time. The χ function simply reads

$$\chi = e^{-\psi} \left((-\partial_t R)^2 + (\partial_r R)^2 \right) \, , \tag{II.32}$$

which leads to the (future trapped) horizon condition

$$(\partial_t R)_H = -(\partial_r R)_H \, . \tag{II.33}$$

The Kodama vector and associated invariant energy are

$$K = e^{-\psi} \left(\partial_r R, -\partial_t R, 0, 0 \right) \, , \tag{II.34}$$

$$\omega = e^{-\psi} \left(-\partial_r R \partial_t I + \partial_t R \partial_r I \right) \, . \tag{II.35}$$

The dynamical surface gravity reads

$$\kappa_H = \frac{1}{2} e^{-\psi_H} \left(-\partial_t^2 R + \partial_r^2 R \right) \Big|_H \, . \tag{II.36}$$

Due to conformal invariance, the HJ equation is the same as in two-dimensional Minkoswki space-time and we may take

$$\partial_+ I = \partial_t I + \partial_r I = 0 \, . \tag{II.37}$$

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Since the null expansion condition leads to $\Delta x^+ = \Delta t + \Delta r = 0$, for outgoing modes we get

$$I = \int_\gamma (dr\partial_r I + dt\partial_t I) = 2\int_\gamma dr\partial_r I \,. \tag{II.38}$$

Furthermore, due to (II.35) and (II.37), one has

$$\partial_r I = \frac{\omega}{e^{-\psi}(\partial_r R + \partial_t R)} \,. \tag{II.39}$$

and we have a pole at the horizon. Making use of near horizon approximation along the null direction, from (II.33) and (II.36), one has $(\partial_r R)_H + (\partial_t R)_H = 0$, $\Delta t + \Delta r = 0$, thus

$$\begin{aligned}
\partial_r R + \partial_t R &= \left. \left(\partial^2_{rr}R - \partial^2_{rt}R - \partial^2_{tt}R + \partial^2_{tr}R \right) \right|_H (r - r_H) + \dots \\
&= 2\kappa_H(r - r_H) + \dots \tag{II.40}
\end{aligned}$$

As a result, making use of Feynman's prescription, one again arrives at equation (II.2).
In the previous computations various choices of signs have been applied in such a way that it may seem they were chosen somewhat ad hoc in order to get the wanted result. This is not so. Once the future sheet of the trapping horizon has been chosen, and the sign of the Kodama vector so determined that it is future directed, no other sign uncertainties will occur for either outgoing or ingoing particles. On the other hand, if there exist a past sheet in the trapping horizon then using the tunneling picture we may as well compute the action along an inward directed[3] curve at the horizon. Then there will be again a non-vanishing imaginary part, but we can interpret it as a small absorption probability.

3 The First Law for Dynamical Black Holes

Here we present, for the sake of completeness, a derivation of a version of the first law. To this aim, let us introduce another invariant in the normal space, related to the stress-energy tensor:

$$T^{(2)} = \gamma^{ij}T_{ij} \,. \tag{III.1}$$

First, let us prove the following invariant relation

$$\kappa_H = \frac{1}{2R_H} + 8\pi R_H T_H^{(2)} \,, \tag{III.2}$$

valid on the dynamical horizon.
We may use the *EFB gauge*, in which $R = r$ and $\chi = C$ and, in this gauge, we have

$$\kappa_H = \frac{1}{2}\partial_r C_H. \tag{III.3}$$

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[3]The ambiguity inherent in this and analogous terms is easily resolved if the manifold is asymptotically flat.

On the other hand, the Einstein equations are very simple in this gauge and we have

$$\frac{1-C}{2} - \frac{r}{2}\partial_r C = -4\pi r^2 T_v^v \,. \tag{III.4}$$

Thus, on the horizon we get

$$\frac{1}{2} - \frac{r_H}{2}\partial_r C_H = -4\pi r_H^2 T_H^{(2)} \tag{III.5}$$

since $T^r{}_{r\,H} = 0$, which leads immediately to (III.2).

However, it is easy to prove the same relation in another coordinate system, for example the
r-gauge. To this purpose, let us introduce the horizon area and the areal volume associated
with the horizon, with their respective differentials:

$$\mathcal{A}_H = 4\pi R_H^2 \,, \qquad d\mathcal{A}_H = 8\pi R_H dR_H \,, \tag{III.6}$$

$$V_H = \frac{4}{3}\pi R_H^3 \,, \qquad dV_H = 4\pi R_H^2 dR_H \,. \tag{III.7}$$

Then a direct calculation gives

$$\frac{\kappa}{8\pi}d\mathcal{A}_H = d\left(\frac{R_H}{2}\right) + T_H^{(2)}dV_H \,. \tag{III.8}$$

In turn, this equation can be recast in the form of a first law, once we introduce the MS
energy at the horizon:

$$dm = \frac{\kappa}{2\pi}d\left(\frac{\mathcal{A}_H}{4}\right) - T_H^{(2)}dV_H \,, \tag{III.9}$$

where $S_H = \mathcal{A}_H/4$ generalizes the Bekenstein-Hawking black hole entropy .

4 An Explicit Example: The FRW Spacetime

A very interesting example of the tunneling method is provided by a generic FRW space-time
with flat spatial sections,

$$ds^2 = -dt^2 + a(t)^2 dr^2 + a(t)^2 r^2 d\Omega^2 \,. \tag{IV.1}$$

At first glance, this example can seem to lie somehow outside the main stream of the paper
which, up to this point, has been devoted to the study of dynamical black hole horizons.
However, what we are considering is a truly dynamical horizon but, this time, of cosmological
interest.

As explained above, metric (IV.1) belongs to the class of Lemaître-Rylov, a subset of the
synchronous dynamical space-times, and we have just showed this to be a coordinate system
where the evaluation of the imaginary part of the action gets a contribution due to the

integration along the t–coordinate.

The normal reduced metric is diagonal with coefficients $\gamma_{ij} = \mathrm{diag}(-1, a(t)^2)$ and $\chi = 1 - r^2 \dot{a}(t)^2$. The dynamical horizon is the Hubble horizon (we assume $H(t) > 0$, see also [25])

$$r_H = \frac{1}{a(t)H(t)} \tag{IV.2}$$

with the Hubble parameter $H(t) = \dot{a}/a$. The dynamical surface gravity is

$$\kappa_H = -\left(H + \frac{\dot{H}}{2H}\right) , \tag{IV.3}$$

and the minus sign refers to the fact the horizon in question (IV.2) is, in Hayward's terminology, of the inner type. For example, in the Einstein-de Sitter model, with $a(t) \propto t^{2/3}$, we would obtain

$$\kappa_H = -\frac{1}{6t}$$

while the de Sitter model in inflationary coordinate has $\dot{H} = 0$ and $\kappa_H = -\sqrt{\Lambda/3}$, though this result is valid in any other patch; furthermore, $\kappa_H = 0$ is only possible in a radiation dominated universe, where $a(t) \propto \sqrt{t}$. One easily sees that in general, for a flat model with $a(t) \propto t^n$ one has

$$\kappa_H = -\left(n - \frac{1}{2}\right) t^{-1}$$

so only for $n < 1/2$ is our surface gravity positive. This regime should not be physically allowed, however, since radiation dominated models occur either with massless particles or with ultra-relativistic massive ones, both of which are limiting cases. It seems as if in these cases we should define the temperature as $T = |\kappa_H|/2\pi$. However, we will see that the tunneling method just gives the right signs without invoking absolute values.

For a massless particle, the reduced HJ equation is

$$\partial_t I = \frac{1}{a(t)} \partial_r I . \tag{IV.4}$$

The full classical action of outgoing particles is

$$I = \int_\gamma dx^i \, \partial_i I , \tag{IV.5}$$

with γ an oriented curve with positive orientation along the increasing values of $x^i = (t, r)$. Radially moving massless particles follow, of course, a null direction. Then, we can perform a null-horizon radial expansion,

$$0 = ds^2 = -dt^2 + a^2(t)dr^2 \tag{IV.6}$$

which gives

$$\Delta t = a(t)\Delta r \,. \tag{IV.7}$$

for out particles. The outgoing particle action, that is the action for particles coming out of the horizon, is then

$$I_+ \;=\; \int dt\, \partial_t I + \int dr\, \partial_r I \tag{IV.8}$$

$$\;=\; 2\int dr\, \partial_r I \tag{IV.9}$$

upon using equation (IV.7). The Kodama vector reads $K = (1, -rH, 0, 0)$ and the invariant energy of a particle is given by

$$\omega = -\partial_t I + rH\partial_r I \,. \tag{IV.10}$$

Thus

$$\partial_r I = \frac{a(t)\omega}{ra(t)H - 1}\,, \tag{IV.11}$$

and we have that

$$I_+ = -2\int dr \frac{a(t)\omega}{1 - ra(t)H(t)}\,. \tag{IV.12}$$

Expanding the function $f(r,t) := 1 - ra(t)H(t)$ close to the horizon, again along a null direction, we get

$$f(r,t) \;\approx\; -aH|_H \Delta r_H - r\ddot{a}|_H \Delta t_H + \dots$$

$$\approx\; 2a(t)\left[-\left(H + \frac{\dot{H}}{2H}\right)\right](r - r_H) + \dots$$

$$\equiv\; 2\kappa_H a(t)(r - r_H) + \dots \,, \tag{IV.13}$$

where κ_H represents the (dynamical) surface gravity associated to the horizon.
The action of the outgoing particle now reads,

$$I_+ = -\int_\gamma dr \frac{\omega}{\kappa_H(r - r_H - i0)}\,. \tag{IV.14}$$

In order to deal with the simple pole in the integrand, we implement the Feynman's $i\epsilon$ - prescription. In the final result, beside a real (irrelevant) contribution, we obtain the following imaginary part:

$$\operatorname{Im} I_+ = -\frac{\pi\omega_H}{\kappa_H}\,. \tag{IV.15}$$

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As a consequence, we may interpret $T = -\kappa_H/2\pi > 0$ as the dynamical temperature associated with FRW flat space-times. In particular, this gives naturally a positive temperature for de Sitter spacetime, a long debated question years ago, usually resolved by changing the sign of the horizon's energy. It should be noted that in the literature, the dynamical temperature is usually given in the form $T = H/2\pi$, with a missing term depending on $\dot{H}$, exceptions being the papers [26]. Again, T becomes negative only for the unphysical flat models with $n < 1/2$, or perhaps we may say there can be no tunneling processes from them.

It is instructive to reconsider the FRW tunneling computation in another coordinate system discussed in previous Sections. Making the coordinate change $R := ra(t)$, the metric assumes the form of r-gauge, namely

$$ds^2 = -(1 - H^2 R^2)dt^2 - 2HR\,dt dR + dR^2 + R^2 d\Omega^2 \tag{IV.16}$$

Note that the metric remains regular on the horizon and the associated normal metric is of the type (II.11). As a result, $\chi = \gamma^{RR}$ and the dynamical horizon is defined by $H^2 R_H^2 = 1$. Of course, the dynamical surface gravity has to remain unchanged with respect to (IV.3). However, in this particular case, the Kodama vector is very simple, $K = (1, \overrightarrow{0})$. As a consequence, the invariant energy is just $\omega = -\partial_t I$, and the HJ equation for a massless particle along a radial trajectory reads:

$$- (\partial_t I)^2 - 2HR(\partial_t I)(\partial_R I) + (1 - H^2 R^2)(\partial_R I)^2 = 0\,. \tag{IV.17}$$

We know already from §II, that the integration along temporal coordinates gives merely a real contribution. Thus, an imaginary contribution to the particle action, comes only from integration along the radial direction. The HJ equation (IV.17) supplemented of the Kodama energy constraint, gives

$$(\partial_R I)_\pm = -\frac{HR\omega \pm \sqrt{(HR\omega)^2 + (1 - H^2 R^2)\omega^2}}{1 - H^2 R^2} \tag{IV.18}$$

Making a null-expansion on the horizon,

$$0 = \Delta s^2 = -2\Delta t \Delta R + \Delta R^2 \tag{IV.19}$$

we see that

$$(1 - H(t)^2 R^2) \approx -2\left(H + \frac{\dot{H}}{2H}\right)\Delta R \equiv 2\kappa_H(R - R_H). \tag{IV.20}$$

We finally get the expected results,

$$\mathrm{Im}\, I_+ = -\int dR \frac{\omega HR\left(1 + \sqrt{1 + O(R - R_H)}\right)}{-2(H + \dot{H}/2H)(R - R_H - i0)} = -\frac{\pi\omega_H}{\kappa_H}\,, \tag{IV.21}$$

with κ_H provided by (IV.20).

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Everything we said above generalizes straightforwardly to models with non-vanishing spatial curvature ($k = 0, \pm 1$), except that the surface gravity is given by the more complicated formula

$$\kappa_H = -\left(H^2 + \frac{1}{2}\dot{H} + \frac{k}{2a^2} \right) R_H \tag{IV.22}$$

where $R_H = (H^2 + k/a^2)^{-1/2}$ is the cosmological trapping horizon, coinciding with the Hubble sphere in the flat case. In this case, one may note that $\kappa_H = 0$ is possible as soon as

$$a(t) = \sqrt{-kt^2 + c_1 t + c_2}\,, \tag{IV.23}$$

where c_1 and c_2 are constants. Of course, for $k = +1$, the solution is real only in a finite range between a big bang and a big crunch.

5 Static Black Holes

Static black hole solutions may be considered as a special case of dynamical ones. However, they are consistent solutions of relativistic theory, Einstein or modified alternative theories, and they deserve a separate analysis, even though the horizon tunneling, which we are going to discuss according to the general procedure outlined in the previous Sections, is a signal of their quantum instability and the static hypothesis we assume is only an approximation, strictly valid only for limited periods of time.

5.1 Kodama-Hayward vs Killing Surface Gravity

Let us consider the Schwarzschild diagonal gauge, which can be written as

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{W(r)} + r^2 d\Omega^2\,. \tag{V.1}$$

In this case the function χ defined in (I.4) coincides with W. Thus, if we assume that V and W have the same simple zeros, we have, on the horizon defined by $\chi = W = 0$,

$$\frac{V_H}{W_H} = \frac{V'_H}{W'_H}\,, \tag{V.2}$$

via de l'Hôspital ratio rule. These coordinates, as is well known, are singular on the horizon, so that the null expansion we have to use is meaningless, the temporal contribution to the action being ill defined. The use of these singular coordinates has been the origin of a large number of papers, containing several proposals to deal with the ambiguity [10]. Recall that in the original paper by Parikh & Wilczek on the tunneling method [3], a clever use of a coordinate system, regular on the horizon, known as Painlevé-Gullstrand coordinates (PG), was advocated. The general PG gauge reads

$$ds^2 = -V dt^2 - 2\sqrt{\frac{V}{W}(1 - W)}dr dt + dr^2 + r^2 d\Omega^2\,. \tag{V.3}$$

According to our assumption (V.2), this gauge is indeed regular on the horizon, being of what we termed r-gauge type. A variant of this gauge is the EF gauge, where an advanced (retarded) time appears, i.e.

$$ds^2 = -V dt^2 - 2\sqrt{\frac{V}{W}} dr dt + r^2 d\Omega^2 \,. \tag{V.4}$$

In both gauges, we may apply the general analysis and conclude that the temporal contribution is absent. The Kodama vector is $K = (\sqrt{\frac{V}{W}}, \overrightarrow{0})$ and the related invariant energy $\omega = \sqrt{\frac{V}{W}} \partial_t I$. One should note that the Kodama vector and the invariant energy do not coincide with the Killing vector and $\partial_t I$ unless $V = W$. This has some consequences, since the general theory gives a surface gravity of

$$\kappa_H = \frac{W'_H}{2} \,, \tag{V.5}$$

instead of the surface gravity associated to the Killing vector ∂_t

$$\kappa_K = \frac{\sqrt{W'_H V'_H}}{2} \,. \tag{V.6}$$

More details can be found in [22], but for now we just limit ourselves to note that the tunneling probability

$$\Gamma \simeq e^{-2\pi \frac{\omega_H}{\kappa_H}} \,, \tag{V.7}$$

is the measurable quantity and since

$$\frac{\omega_H}{\kappa_H} = \frac{E}{\kappa_K} \,, \tag{V.8}$$

where $E = \partial_t I$ is the Killing energy, no contradiction within the static approximation can be found. However, as soon as the space-time becomes dynamic, a Killing vector is useless and the only invariant energy is ω, indeed.

5.2 Dilaton-Maxwell-Einstein Black holes

In this Subsection, we would like to give a brief review to black hole static solutions in the form (V.1), with $W \neq V$.

As far as one is concerned with Einstein's GR, solutions of this kind are forbidden by well-known uniqueness theorems, cfr. [28]. However, it is not difficult to face them in alternative theories of gravity, a label which actually contains a lot of different material. In order to be more precise, let us consider the specific example provided by the so-called Dilaton-Maxwell-Gravity (DMG) as in [13] and [29, 30, 31, 32]. We typically start by an action such as [32],

$$I = \int d^4 x \sqrt{-g} \left(R - 2(\nabla\phi)^2 - V(\phi) - e^{-2\xi\phi} F^2 \right) \tag{V.9}$$

where R is the scalar curvature, $F^2 = F_{ab}F^{ab}$ and ξ governs the coupling of the dilaton with the Maxwell field. Varying the action (V.9) with respect to the metric, Maxwell and dilaton fields yields the EOM for the respective fields. It is easier if we consider Maxwell fields generated by an isolated electric charge. Then, the ansatz

$$ds^2 = -U(r)dt^2 + U^{-1}(r)dr^2 + H^2(r)d\Omega^2 \tag{V.10}$$

for the line element, satisfies the EOM for the metric under very general conditions.[4]

Assuming $b < a$, a first class of solutions is given by [30]

$$U(r) = \frac{1 - \frac{a}{r}}{1 - \frac{b}{r}} \quad \& \quad H(r) = r^2 - br \,, \tag{V.11}$$

another one is instead

$$U(r) = 1 - \frac{a}{r} \,, \quad \& \quad H(r) = r^2 - br \,. \tag{V.12}$$

For a quite general class of solutions, but without any attempt of universality, the $U(r)$ function in (V.10) can be expressed in terms of $H(r)$ and the dilaton as

$$U(r) = \frac{\xi r + \eta}{H^2(r)\phi'(r) + \xi H(r)H'(r)}. \tag{V.13}$$

where η is an integration constant and the prime denotes derivation with respect to the argument.

Next, consider a conformal transformation

$$d\tilde{s}^2 = \Omega^2 ds^2 \tag{V.14}$$

so that

$$\Omega^2(r) = \frac{r^2}{H^2(r)}. \tag{V.15}$$

The conformal metric looks like (V.1),

$$d\tilde{s}^2 = -V(r)dt^2 + W^{-1}(r)dr^2 + r^2 d\omega_2^2 \tag{V.16}$$

with

$$V(r) = \frac{r^2 U(r)}{H^2(r)} = \frac{r^2(\xi r + \eta)}{H^4 \omega + \xi H^3 H'} \tag{V.17}$$

$$W(r) = \frac{U(r)H^2(r)}{r^2} = \frac{H(\xi r + \eta)}{r^2(H\omega + \xi H')}. \tag{V.18}$$

If the spacetime possesses a horizon, this will be located at r_0, s.t. $W(r_0) = 0$, i.e. where

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[4]The Maxwell field then reads $F_{tr} = \frac{Qe^{2\xi\phi}}{H^2(r)}$, Q being the electric charge.

1. $U(r_0) = 0$, that is $r_0 = -\eta/\xi > 0$;

2. $H(r_0) = 0$. Note that in this case

$$\lim_{r \to r_0} V(r) = \infty \qquad \lim_{r \to r_0} \Omega^2(r) = \infty , \tag{V.19}$$

   that is, the conformal transformation becomes singular.

It is simply a question of algebra to compute the Kodama-Hayward surface gravity and see that it is well-defined in both cases 1 and 2. But, with regard to the Killing surface gravity, we have to distinguish carefully between the two cases under examination. Indeed, case 1 is easy to treat and gives $\kappa_K = U_0'/2$: something we had to expect from the very beginning in consideration of the well-established result according to which the Killing Hawking temperature $\Theta = \kappa_K/2\pi$ is conformally invariant [33]. But things go radically different in case 2 of a singular conformal transformation, where the same definition of a Killing surface gravity, positive and finite on the horizon becomes, at least in the general case, questionable.

This analysis, however, is sufficient to shed new light on the stringy black hole puzzle. Start in fact from the metric (V.10) with, for example, $U(r) = 1 - a/r$, $a = const$ and $H(r) = \sqrt{r(r-b)}$, $b < a$. Perform a conformal transformation in order to get the GHS solution [30]. Since $b < a$, $r_0 = a$ is still an event horizon. The Killing surface gravity, being invariant under conformal transformation, does not feel the new physics introduced by the conformal transformation. Thus no change has to be expected in the extremal limit. The story is different for Hayward's surface gravity that goes like $\kappa_H \propto H_0^2$ and vanishes whenever the conformal factor vanishes, e.g. in the extremal limit for GHS solution.

5.3   The Lemaître-Rylov Gauge

As further example, let us consider the Schwarzschild spacetime in coordinates (t, r, θ, φ) such that the line element can be expressed as

$$ds^2 = -dt^2 + \frac{dr^2}{B} + (r_g B)^2 d\Omega^2 , \tag{V.20}$$

where $r_g = 2m$ is the usual gravitational radius, and

$$B(t,r) := \left[\frac{3}{2r_g}(r-t) \right]^{\frac{2}{3}} . \tag{V.21}$$

We shall refer to these coordinates as the Lemaître-Rylov gauge. This is indeed an interesting (time-dependent) gauge since - contrary for example to isotropic coordinates - (t, r) extend beyond the gravitational radius, $r < r_g$. It is an explicit example of synchronous gauge. Notice further that [28]:

1. The spacetime singularity is located at $r = t$;

2. The horizon is located in correspondence of

$$B(t,r)\big|_H = 1, \quad (r_H - t) = \frac{2r_g}{3}. \tag{V.22}$$

3. Outgoing particles are such that $\frac{dt}{dr} < 0$; ingoing particles instead have $\frac{dt}{dr} > 0$ (cfr. spacetime diagram in [28] for example).

A detailed calculation can now be provided.
According to the general procedure outlined above,

$$\gamma_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & B^{-1} \end{pmatrix}, \quad \sqrt{-\gamma} = B^{-1/2}, \tag{V.23}$$

and

$$\gamma^{ij} = \begin{pmatrix} -1 & 0 \\ 0 & B \end{pmatrix}. \tag{V.24}$$

Making use of (I.7), a direct computation leads to $k_H = \frac{1}{2r_g}$, as expected. The explicit form of the Kodama vector K in terms of the Lemaître gauge is

$$K = (1, 1, 0, 0), \tag{V.25}$$

so that the particle's energy is

$$\omega = -\partial_t I - \partial_r I. \tag{V.26}$$

The H-J equation for radially moving particles is

$$-(\partial_t I)^2 + B(\partial_r I)^2 = 0, \tag{V.27}$$

and for the outgoing particles, we take

$$\partial_r I = -\frac{\partial_t I}{\sqrt{B}}. \tag{V.28}$$

Thus

$$\partial_r I = \frac{\omega(\sqrt{B} + 1)}{B - 1}. \tag{V.29}$$

The null expantion condition gives for the outgoing particle $\Delta t = -\frac{\Delta r}{\sqrt{B}}$, namely we have

$$I = 2 \int_\gamma dr \, (\partial_r I)_+ = 2 \int_\gamma dr\omega \, \frac{\sqrt{B} + 1}{B - 1}. \tag{V.30}$$

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Let us define $\mathscr{A}(t, r) := B - 1$ and expand it along the null, radial, outgoing geodesic close to the horizon:

$$B - 1 \simeq \partial_r B_H \Delta r + \partial_t B_H \Delta t = 2\partial_r B_H \Delta r = \frac{2}{r_g}(r - r_H). \tag{V.31}$$

As a result

$$I = r_g \int_\gamma \frac{\omega(\sqrt{B} - 1)}{(r - r_H - i0)} \quad \Longrightarrow \quad Im\, I = \frac{\pi\omega}{k_H} = 4\pi m\omega \tag{V.32}$$

which provides the correct Hawking temperature from the hole. The Lemaître-Rylov gauge can be generalized beyond Schwarzschild spacetime. In general the metric we shall deal with is

$$ds^2 = -dt^2 + (1 - V(R))dr^2 + R^2 d\Omega^2, \tag{V.33}$$

where $V(R)$ and the areal radius R are function of $r \pm t$, via the inversion of the relation

$$r \pm t = G(R) = \mp \int dR \sqrt{\frac{1}{WV}} \left(\sqrt{\frac{1}{1 - V}} - \sqrt{1 - V} \right). \tag{V.34}$$

We are dealing with a dynamical synchronous gauge and we may apply the general formalism seen above. A straightforward calculation leads again to (V.5) [5].

6   Conclusions

In the last few years, many different proposals have been suggested in order to give a universal prescription for the Hawking temperature of certain dynamical spacetimes, especially those endowed with future trapping horizons. In such cases, for example, one lacks the Kubo-Martin-Schwinger (KMS) condition so successful for equilibrium states. Nor is there generally available a analytic continuation to Euclidean signature within which one can judge the periodicity of Euclidean time, essentially equivalent to the KMS condition. In fact, one can even doubt that a temperature with the usual meaning is generally possible or useful at all. The final result is that time by time many prescriptions were proposed which either were not applicable to all the desired cases one can have in mind, or simply were invented ad hoc to keep the peace with the most well known cases. For example, for a slowly varying Schwarzschild black hole it seemed natural to keep the temperature equal to the instantaneous value $1/8\pi M(t)$ characterizing the static black hole, although this is not so obvious.

On the other hand we are all accustomed to the remarkably universal properties exhibited by black holes so, moved by the wish of extending some of these to a dynamical regime, we

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[5] As an exercise, the interested reader could be curious to see what happens by introducing Kruskal-Szekeres generalized coordinates for the metric (V.1). The computation is rather long and tedius, but still confirms, as it must be, the general covariance of Hayward's surface gravity.

have written this note in order to clarify (hopefully) the so-called HJ method for horizon tunneling.

The playground has been given by the class of spherically symmetric spacetimes, either static or dynamical, and within this class we can draw the conclusion that in order for the HJ method to work properly: (i) regular coordinates at the horizon are necessary; (ii) an appropriate notion of particle energy, that it should be a scalar, can be implemented at the level of HJ equation in such a way that imaginary contributions to particle action of outgoing particles arise after a near-horizon approximation; (iii) we have shown the relevance of the null-expansion of (to all purposes, massless) particles through the horizon in order to reconstruct fully the particle action. In turn, it is the particle action responsible for the particle production rate (I.1).

As a result, it should also be clear that a contribution to the rate from the temporal part of the integration, that is from the piece $\int_\gamma \partial_t I dt$ of the basic tunneling rate, is generally present but gauge dependent; it is only the full integration along $\overrightarrow{\gamma}$ that produces a gauge invariant result. So for example, in certain gauges there is a temporal contribution, in some other gauge there is not.

Finally, in static spacetimes violating the weak energy condition it seems that the choice between Killing and Kodama-Hayward cannot be decided at the formal level, although it may be remarked that the difference amounts only to a trivial scaling, albeit one which can have less trivial effects on extremal black holes. It may also be noted that, as soon as a static black hole starts evaporating, the spacetime ceases to be rigorously static and only the dynamical picture survives, a picture where the Killing vector with its associated energy/surface gravity/temperature plays a much more minor role.

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